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Stability of equilibrium in a one-dimensional hydrostatic model of the dry atmosphere

S.M. Semenov

Yu. A. Izrael Institute of Global Climate and Ecology,
20B, Glebovskaya str., 107258, Moscow, Russia;
Correspondence address: SergeySemenov1@yandex.ru

Institute of Geography of the Russian Academy of Sciences,
29, Staromonetny per., 119017, Moscow, Russia,

National Research University Higher School of Economics,
20, Myasnitskaya str., 101000, Moscow, Russia

Abstract. Experts in the field of mathematical modeling of the climate system have different views on the class of models that should be employed to analyze and predict climate for time scales corresponding to climatic processes. In this paper, we investigate the properties of a model constructed using the hydrostatic hypothesis. A one-dimensional (horizontally homogeneous) hydrostatic model of a dry atmosphere is considered. Air is considered an ideal gas. The source of heat is the external short-wave radiation flux entering the upper boundary of the atmosphere. This energy is partly absorbed by atmospheric layers and the underlying surface, partly returned to space. The atmospheric layers and the underlying surface radiate in the long-wave range. In general, the absorption coefficient and heat capacity are specific for the atmospheric layers and are everywhere positive. In the model, the radiation balance of a segment of the atmospheric column above a unit area of the underlying surface determines the change in the internal energy and the volume occupied by the segment. The pressure value always remains equal to the weight of a part of the atmospheric column above the segment (hydrostatic hypothesis). The underlying surface is always in the state of radiation equilibrium. Under these assumptions: a) there is a single equilibrium vertical temperature distribution in the column and corresponding air pressure and density distributions (they are calculated using the hydrostatic assumption and the equation of a state of the ideal gas); b) the temperature distribution is asymptotically stable, i.e. any other initial distribution of non-negative temperature values tends with time to equilibrium uniformly on the vertical. Thus, one can expect that the numerical analogs of the model considered in this work will also be stable, which is important for the computational implementation of both the one-dimensional model and its three-dimensional versions.

Keywords. Dry atmosphere, one-dimensional model, hydrostatic hypothesis, equilibrium state, asymptotic stability.

Introduction

One-dimensional, horizontally homogeneous model of the atmosphere is a subject of purely theoretical studies. Using this model, one can only clarify the understanding of the main processes taking place in the real atmosphere. In relation to the Earth's atmosphere, among Russian scientists, similar studies were conducted by A.A. Fridman (Friedmann, 1914) using the simplest version of the Navier-Stokes equations; see also the memorial collection of articles (Fridman, 1966).

In this paper the following simplified model concepts of dynamics of a one-dimensional atmosphere are used.

I. The total mass (kg) of air M_0 in the atmospheric column above the area of 1 m^2 is finite and constant; its distribution $\rho(t, z)$ along the height z ($z = 0$ corresponds to the underlying surface) at any moment t (s) is positive everywhere.

II. At any moment t, the pressure $P(t, z)$ (Pa) at a point at the height z is equal to

$$g \int_z^{+\infty} \rho(t,\xi)\, d\xi,$$

(where g is the acceleration due to gravity, 9.8 m s^{-2}), i.e. to the weight of air column over the unit of 1 m^2 at the height z (hydrostatic hypothesis). At altitudes up to 100 km above sea level, i.e. up to the generally accepted conditional altitude boundary of the Earth's atmosphere, the acceleration of gravity can be considered constant with satisfactory accuracy for theoretical studies.

R.I. Nigmatulin (2018) believes that the hydrostatic hypothesis is applicable when calculating atmospheric processes at significant times, at which, on average, the vertical components of inertial forces are substantially less than the gravitational force. Although there is some disagreement among specialists on this subject, the hydrostatic hypothesis is adopted in this study.

III. The vertical distribution of air density (kg m^{-3}) $\rho(t, z)$ evolves over time in such a way that

$$\int_{z(t)}^{+\infty} \rho(t,\xi)\, d\xi,$$

(where $z(t)$ is the trajectory of any given point) remains constant; thus, the pressure value remains unchanged, see item II.

This means, in particular, that for each t one can use a replacement of variables , under which segments along the height $[z_1, z_2]$, $z_1 < z_2$, will uniquely correspond to certain segments by mass $[m_1, m_2]$, $m_1 < m_2$, and *vice versa*. Such a replacement of variables is widely used (see, for example, (Belov, 1975)). In this case, for small segments $\Delta z = z_2 - z_1$ and $\Delta m = m_2 - m_1$, the relation $\Delta m = \rho\, \Delta z$ takes place, accurate to infinitesimals of higher order of smallness.

This property of the density evolution in time also takes place in the classical one-dimensional Navier-Stokes model, the system of equations of which includes the continuity equation

$$\frac{\partial \rho(t,z)}{\partial t} = -\frac{\partial}{\partial z}\big(v(t,z)\rho(t,z)\big).$$

Indeed, under the assumption that on the underlying surface ($z = 0$) *the vertical air velocity* $v(t, 0)$ *is always zero*

$$\frac{d}{dt}\int_0^{z(t)} \rho(t,\xi)\,d\xi = \int_0^{z(t)} \frac{\partial \rho(t,\xi)}{\partial t}\,d\xi + v\big(t,z(t)\big)\,\rho\big(t,z(t)\big) =$$

$$-\,v\big(t,z(t)\big)\,\rho\big(t,z(t)\big) + v(t,0)\,\rho(t,0) + v\big(t,z(t)\big)\,\rho\big(t,z(t)\big) = 0.$$

This, in particular, means that there is no vertical mixing in the model. If at some moment t_0 $z_2(t_0) \geq z_1(t_0)$, *then the relation* $z_2(t) \geq z_1(t)$ *will be valid also later for* $t > t_0$.

IV. For any small segment on the mass scale $[m_1, m_2]$, $\Delta m = m_2 - m_1 > 0$, and small $\Delta t > 0$, accurate to infinitesimals of higher order smallest, the change in temperature T of the segment of masses $[m_1, m_2]$ satisfies the relation

$$c_V(m_1)\Delta m\,\Delta T + P\Delta V = q\,\Delta m\,\Delta t\,. \qquad (1)$$

In this formula: c_V is the heat capacity of air of a given segment at a constant volume; V is the length of the height segment corresponding to the segment of masses $[m_1, m_2]$; ΔV is its change during the time Δt; q is radiation balance in the segment of masses $[m_1, m_2]$ per unit mass and unit of time. The net input of radiant heat $q\,\Delta m\,\Delta t$ is spent only on changes in volume ΔV and temperature ΔT of mass Δm.

V. The only primary source of heat in the model is the short-wave radiation flux entering the upper boundary of the atmosphere. This radiation can be absorbed to some extent by the atmospheric layers and the underlying surface, and is also directed back into space due to reflection by the underlying surface and scattering by the atmospheric layers.

VI. The atmospheric layers and the underlying surface radiate in the long-wavelength range. In the model, the underlying surface completely absorbs the long-wave radiation of the atmospheric layers that arrives at it. The underlying surface is in radiation equilibrium, i.e. the downward total flux of radiation from the atmosphere is always equal to the upward flux of long-wave radiation from the underlying surface.

The equation of state of an ideal gas for dry atmospheric air for a height segment $[z_1, z_2]$ gives the relation

$$PV = \left(\frac{R}{\mu}\right)\Delta m\,T,$$

where $R = 8.314$ J mol$^{-1} \cdot$ K^{-1} is the gas constant, while $\mu = 0.029$ kg mol^{-1} is the molecular mass of air.

Moving to the increments in the left and right sides, taking into account the invariability of mass Δm and the pressure P (see item II) up to small values of a higher order of magnitude (see item III), we obtain the relation

$$P\,\Delta V = \frac{R}{\mu}\,\Delta m\,\Delta T.$$

Combining this relation with relation (1), we obtain the following equation as $\Delta t \to 0$:

$$\frac{\partial T(t,m_1)}{\partial t} = \frac{1}{c_P(m_1)}\,q(t,m_1). \qquad (2)$$

Here: $T(t, m_1)$ is the absolute air temperature (K) at such a point at a height z, the air mass below which in the atmospheric column above the underlying surface unit of 1 m^2 is equal to m_1 kg; $c_P(m_1)$ is the heat capacity of air at this point at constant pressure; $q(t, m_1)$ is the radiation balance at this point per unit mass and unit time. It depends on the descending and ascending fluxes of radiation, which, in turn, are determined by fluxes of short-wave (solar) radiation, fluxes of long-wave radiation of atmospheric layers and the underlying surface, as well as by the absorption coefficients of atmospheric layers. These fluxes are detailed in the **Results** section.

Thus, equation (2) determines the evolution of the temperature profile $T(t, m_1)$ with time. Taking into account the assumption of hydrostaticity

$$P(t,\,m_1) = g(M_0 - m_1), \qquad (3)$$

the evolution of air density profile with time can be found from the equation of state of the ideal gas $P = \frac{R}{\mu}\,\rho\,T$:

$$\rho(t,m_1) = \frac{\mu}{R\,T(t,m_1)}\,g(M_0 - m_1). \qquad (4)$$

The height z_1 corresponding to the mass m_1 is

$$z_1 = \int_0^{m_1} \frac{dm}{\rho(t,m)}\,.$$

Relations (2), (3) and (4) describe the evolution with time of the profiles, respectively, of temperature, pressure and density of atmospheric air in the model. The existence and stability of the equilibrium state in this model is determined completely by these properties for equation (2).

The purpose of this work is to investigate the properties of equation* (2), *namely: − to describe its stationary solution, which characterizes the equilibrium state; − to investigate stability of this state.

Methodological remarks

To analyze equation (2) for stability, we need two technical lemmas given in this methodological section. They relate to the behavior of solutions of differential equations for functions whose argument is a real variable, and its values are real-valued functions on a segment. On the right-hand side of these differential equations, there is a linear integral operator transforming the function. Such operators and properties of systems the evolution of which in time is described using these operators are presented, e.g., in (Krasnosel'skiy, 1962; Integral equations, 1968; Positive linear systems, 1985).

The goal of this methodological section is quite particular, namely, to unload from technical details the presentation of the study of stationary solution of equation (2) and its stability, which is given in the next section **Results**. Therefore, they are combined in the two technical Lemmas given in this methodological section. One can skip this section and continue reading from the **Results** section, and then return to this section for technical details, if necessary.

For the function f below, the symbols f_- and f_+ denote the non-positive and non-negative parts of f, i.e. min$\{-f, 0\}$ and max$\{f, 0\}$, respectively. The symbol ■ means the end of proof.

Lemma 1. Let $f(t, m)$ be a continuous function on $[t_0 , +\infty]$ x $[0, M_0]$ differentiable in terms of t, $\frac{\partial f(t,m)}{\partial t}$ is also continuous, and $f(t, m)$ satisfies the equation

$$\frac{\partial f(t,\ m)}{\partial t} = g(t,m)\left\{\int_0^{M_0} k(m,m_1)f(t,m_1)dm_1 - f(t,m)\right\},$$

where $g(t, m)$ is a continuous function on $[t_0 , +\infty]$ x $[0, M_0]$ taking only positive values, $k(m, m_1)$ is a continuous function on $[0, M_0]$ x $[0, M_0]$ which takes only non-negative values, and there exists $0 < C < 1$ such that $\int_0^{M_0} k(m,m_1)dm_1 < C$ for any $m \in [0, M_0]$.

Then

a) the function $\varphi(t) = \sup_{m\,\epsilon[0,M_0]} f_+(t,m)$ is continuous, and for any $t_* \geq t_0$, $t^* > t_*$, if $\varphi(t_*) > 0$ then $\varphi(t^*) < \varphi(t_*)$, and $\varphi(t_*) = 0$ implies $\varphi(t^*) = 0$;

b) the function $\psi(t) = \inf_{m\,\epsilon[0,M_0]} f_-(t,m)$ is continuous, and for any $t_* \geq t_0$, $t^* > t_*$, if $\psi(t_*) < 0$ then $\psi(t^*) > \psi(t_*)$, and $\psi(t_*) = 0$ implies $\psi(t^*) = 0$.

Proof. Since assertion b) is equivalent to assertion a) when the function f is replaced by $(-f)$, it suffices to prove assertion a).

1. By the condition of the Lemma for the functions f, g and k, the modulus of the right-hand side of the equation and, hence, the left-hand side $\frac{\partial f(t,m)}{\partial t}$ is uniformly bounded on any $[t_1, t_2]$, $t_1, t_2 \geq t_0$. Therefore, on the segment $[t_1, t_2]$, the function $f(t, m)$ is equicontinuous in terms of t. Since $f_+ = (|f| + f)/2$, the same holds for f_+.

The following holds for any t_*, $t^* \geq t_0$:

$$|\varphi(t^*) - \varphi(t_*)| = |\sup_{m\,\epsilon[0,\,M_0]} f_+(t^*,m) - \sup_{m\,\epsilon[0,\,M_0]} f_+(t_*,m)| \leq .$$

$$\leq \sup_{m\,\epsilon[0,\,M_0]} |f_+(t^*,m) - f_+(t_*,m)| .$$

The expression on the right-hand side of the inequality tends to zero as $t^* \to t_*$ due to equicontinuity of the function f_+ in terms of t.

2. Let $t > t_0$, $\varphi(t) = a > 0$, and X is a subset of such $m \,\epsilon\, [0, M_0]$ that $f(t, m) = a$. Since f is a continuous function, X is compact. Let us show that for any $m \,\epsilon\, X$ the inequality $\frac{\partial f(t,m)}{\partial t} < 0$ holds. Indeed,

$$\frac{\partial f(t,\ m)}{\partial t} = g(t,m)\left\{\int_0^{M_0} k(m,m_1)f(t,m_1)dm_1 - f(t,m)\right\}$$

$$\frac{\partial f(t,\ m)}{\partial t} = a\ g(t,m)\left\{\int_0^{M_0} k(m,m_1)[f(t,m_1)/a]dm_1 - 1\right\} \leq$$

$$a\ g(t,m)\left\{\int_0^{M_0} k(m,m_1)dm_1 - 1\right\} \leq\ a\ g(t,m)\ (C-1) < 0.$$

Due to continuity of f and $\frac{df}{dt}$, for any $m \,\epsilon\, X$ there exist such a neighborhood U_m and $\Delta_m t > 0$ that $\frac{df}{dt} < b_m$ for some $b_m < 0$ and $f > 0$ on U_m x $[t, t + \Delta_m t]$.

Since X is compact, and a family of the open sets U_m is its cover, one can choose a finite subcover from it. Let U is the union of this finite collection, Δt is the corresponding minimum of $\Delta_m t$, and $b < 0$ is the corresponding maximum of b_m .

We denote the complement of U in $[0, M_0]$ by Y.

Suppose the set Y is not empty. It is compact and have no intersection with the compact set X. Therefore $a_1 = \sup_{m\,\epsilon\,Y} f(t,m) < a$. Since the function f is continuous, for small $\delta t > 0$ the following holds:

$$\sup_{m\,\epsilon\,Y} f(t+\delta t,m) < \sup_{m\,\epsilon\,X} f(t+\delta t,m) \leq \sup_{m\,\epsilon\,U} f(t+\delta t,m).$$

That is, for the value of the first argument $(t + \delta t)$, the least upper bound of the values of the function f in terms of the second argument on the whole $[0, M_0]$ is the same as on the set U.

It remains to note that for small positive $\delta t < \Delta t$

$$0 < \sup_{m\,\epsilon\,U} f(t+\ \delta t,m) < a + b\ \ \delta t < a$$

by the construction of the set U.

If Y is empty, then the last remark suffices.

Thus, it has been shown that if $\varphi(t) > 0$ then $\varphi(t_1) < \varphi(t)$ for small $(t_1 - t) > 0$.

3. Let $t^* > t_* \geq t_0$ and $\varphi(t_*) > 0$. Let us show that $\varphi(t^*) < \varphi(t_*)$.

3.1. Suppose $\varphi(t^*) > \varphi(t_*)$. We denote by t_2 the least upper bound of such $t \in [t_*, t^*]$ that $\varphi(t) \leq \varphi(t_*)$. Since φ is continuous, $\varphi(t_2) = \varphi(t_*)$ and $t_2 < t^*$. Then for any $t \in [t_2, t^*]$ the inequality $\varphi(t) > \varphi(t_2)$ holds, giving a contradiction with item 2 for small $(t - t_2) > 0$.

3.2. Suppose that $\varphi(t^*) = \varphi(t_*)$. Then, due to continuity of function φ:

– there exists such $t \in [t_*, t^*]$ that $0 < \varphi(t_2) < \varphi(t_*)$, giving a contradiction with item 3.1;

– or there exists such $t \in [t_*, t^*]$ that $\varphi(t_2) > \varphi(t_*) > 0$, also giving a contradiction with item 3.1;

– or $\varphi(t) \equiv \varphi(t_*)$ on $[t_*, t^*]$, which is also impossible, because $\varphi(t_*) > 0$ and for small $(t - t_*) > 0$ the inequality $\varphi(t) < \varphi(t_*)$ should hold due to item 2.

Thus, whenever $t^* > t_* \geq t_0$ and $\varphi(t_*) > 0$ the inequality $\varphi(t^*) < \varphi(t_*)$ holds.

4. Suppose $t^* > t_* \geq t_0$ and $\varphi(t_*) = 0$. Let us show that $\varphi(t^*) = 0$. Indeed, if $\varphi(t^*) > 0$ then, due to continuity of function φ, there exists such $t_2 \in [t_*, t^*]$ that $0 < \varphi(t_2) < \varphi(t^*)$, giving a contradiction with item 3.1.

■

Corollary from Lemma 1. Under the conditions of Lemma 1, for any $t \geq t_0$, the following holds:

$$\sup_{m \in [0, M_0]} |f(t,m)| \leq \sup_{m \in [0, M_0]} |f(t_0,m)|.$$

Proof. This follows from the relation

$$\sup_{m \in [0, M_0]} |f(t,m)| = \max\{\varphi(t), -\psi(t)\}.$$

The function presented in the right part of this equality is non-increasing, since it is maximum of two non-increasing functions. ■

Remark to Lemma 1. No properties of the segment $[0, M_0]$, except for the presence of measure and compactness, were used in the proof of Lemma 1. Thus, it is valid not only for the segment $[0, M_0]$, but also for any compact with measure.

Lemma 2. Let $f(t, m)$ be a continuous function on $[t_0, +\infty)$ x $[0, M_0]$ differentiable in terms of t, $\frac{\partial f(t,m)}{\partial t}$ is also continuous, and $f(t, m)$ satisfies the equation

$$\frac{\partial f(t,m)}{\partial t} = g(t,m)\left\{\int_0^{M_0} k(m,m_1) f(t,m_1)dm_1 - f(t,m)\right\},$$

where $g(t, m)$ is a continuous function on $[t_0, +\infty]$ x $[0, M_0]$ such that $g(t, m) > C_0$ for some $C_0 > 0$ over the entire realm of definition, $k(m, m_1)$ is a continuous function on $[0, M_0]$ x $[0, M_0]$, which takes only non-negative values over the entire realm of definition, and there exists such $0 < C < 1$ that $\int_0^{M_0} k(m,m_1)dm_1 < C$ for any $m \in [0, M_0]$.

Then

a) the function $\varphi(t) = \sup\limits_{m \epsilon [0, M_0]} f_+(t,m)$ is non-increasing and tending to zero as $t \rightarrow +\infty$;

b) the function $\psi(t) = \inf\limits_{m \epsilon [0, M_0]} f_-(t,m)$ is non-decreasing and tending to zero as $t \rightarrow +\infty$.

Proof. Since assertion b) is equivalent to assertion a) when the function f is replaced by $(-f)$, it suffices to prove assertion a).

If $\varphi(t_0) = 0$ then $\varphi(t) \equiv 0$ due to Lemma 1, i.e. the assertion a) holds.

Let $\varphi(t_0) > 0$. Due to Lemma 1, the non-negative function $\varphi(t)$ is non-increasing over $t \geq t_0$. Suppose that $A = \lim\limits_{t \to +\infty} \varphi(t) > 0$. Choose $\varepsilon > 0$ such that for any m $[0, M_0]$

$$\frac{(A+\varepsilon)}{A} \int_0^{M_0} k(m, m_1) dm_1 < 1.$$

Any positive $\varepsilon < A(1/C - 1)$ is suitable for this, since $\int_0^{M_0} k(m, m_1) dm_1 < C < 1$ for any $m \epsilon [0, M_0]$ due to conditions of this Lemma.

Choose $t_2 \geq t_0$ such that for any $t \geq t_2$ the inequality $\varphi(t) \leq A + \varepsilon$ holds. For any $t \geq t_2$, we denote by X_t a subset of $[0, M_0]$ over which $f(t, m) \geq A$. This is a collection of embedded closed sets, i.e. $X_{t+\Delta t} \subseteq X_t$ for any $\Delta t \geq 0$. Indeed, if some m_1 satisfies the conditions $m_1 \notin X_t$ and $m_1 \epsilon X_{t+\Delta t}$, then there exists δt, $0 < \delta t \leq \Delta t$, such that for any $t_1 \epsilon [t, t + \delta t)$ the relations $f(t_1, m_1) < A$ and $f(t + \delta t, m_1) = A$ would hold. However, this is not possible, because

$$\frac{\partial f(t+\delta t,\ m_1)}{\partial t} = g(t+\delta t,\ m_1)\left\{\int_0^{M_0} k(m_1, m_2) f(t+\delta t, m_2) dm_2 - f(t+\delta t, m_1)\right\} = g(t+\delta t,\ m_1)\left\{\int_0^{M_0} k(m_1, m_2) f(t+\delta t, m_2) dm_2 - A\right\} =$$
$$A\, g(t+\delta t,\ m_1)\left\{\frac{1}{A}\int_0^{M_0} k(m_1, m_2) f(t+\delta t, m_2) dm_2 - 1\right\} \leq$$
$$A\, g(t+\delta t,\ m_1)\left\{\frac{(A+\varepsilon)}{A}\int_0^{M_0} k(m_1, m_2) dm_2 - 1\right\} < 0$$

due to selection of ε and positivity of the function g.

Thus, the sets X_t for $t \geq t_2$ form a collection of embedded sets. Since these sets are closed subsets of the compact $[0, M_0]$, they have a point $m_* \epsilon [0, M_0]$ in common.

The values of $f(t, m_*)$ for $t \geq t_2$ are within the segment $[A, A + \varepsilon]$. However, this is not possible as $t \rightarrow +\infty$, because

$$\frac{\partial f(t,\ m_*)}{\partial t} = g(t,\ m_*)\left\{\int_0^{M_0} k(m_*, m_2) f(t, m_2) dm_2 - f(t, m_*)\right\} \leq$$
$$g(t,\ m_*)\left\{\int_0^{M_0} k(m_*, m_2) f(t, m_2) dm_2 - A\right\} =$$
$$A\, g(t,\ m_*)\left\{\frac{1}{A}\int_0^{M_0} k(m_*, m_2) f(t, m_2) dm_2 - 1\right\} \leq$$
$$A\, g(t,\ m_*)\left\{\frac{(A+\varepsilon)}{A}\int_0^{M_0} k(m_*, m_2) dm_2 - 1\right\} \leq$$
$$AC_0 \left\{\frac{(A+\varepsilon)}{A}\int_0^{M_0} k(m_*, m_2) dm_2 - 1\right\} < 0\ ,$$

given the choice of ε, and since the function g is bounded below by a positive constant C_0 .

This contradiction shows that if $\varphi(t_0) > 0$ then $\lim_{t \to +\infty} \varphi(t) = 0.$

■

Corollary from Lemma 2. Under the conditions of Lemma 2 $\sup_{m \in [0, M_0]} |f(t,m)|$ is non-increasing and tending to zero as $t \to +\infty$.

Proof. This follows from the relation

$$\sup_{m \in [0, M_0]} |f(t,m)| = \max\{\varphi(t), -\psi(t)\}.$$

Both functions presented in the right-hand part of this equality are non-increasing and tending to zero as $t \to +\infty$ due to Lemma 2. ■

Remark to Lemma 2. No properties of the segment $[0, M_0]$, except for the presence of measure and compactness, were used in the proof of Lemma 2. Thus, it is valid not only for the segment $[0, M_0]$, but also for any compact with measure.

Results

Equilibrium

Let us introduce the following notations:

- $J^{\downarrow}(m)$ and $J^{\uparrow}(m)$ denote, respectively, the descending and ascending fluxes of long-wave radiation at a point at height z, the air mass under which in the atmospheric column above the underlying surface area of 1 m^2 is equal to m;
- $w(m) > 0$ is the absorption coefficient of long-wave radiation at this point per unit mass of air 1 kg over the area;
- $s(m) \geq 0$ is the flux of short-wave radiation absorbed at this point per unit mass of air 1 kg over the area;
- $s_0 > 0$ is the short-wave radiation flux absorbed by the underlying surface area of 1 m^2.

The value s_0 and the profile $s(m)$ are input information for this model and can be calculated, for example, following the algorithm described in (Semenov, Popov, 2011).

Under these assumptions, in a state of equilibrium, the following relations are fulfilled:

$$\frac{dJ^{\uparrow}(m)}{dm} = -w(m)J^{\uparrow}(m) + 0.5[w(m)\left(J^{\uparrow}(m) + J^{\downarrow}(m)\right) + s(m)]$$
$$\frac{dJ^{\downarrow}(m)}{dm} = w(m)J^{\downarrow}(m) - 0.5\left[w(m)\left(J^{\uparrow}(m) + J^{\downarrow}(m)\right) + s(m)\right]. \tag{5}$$

Also, in a state of equilibrium, the ascending flux of long-wave radiation from an atmospheric layer with air mass m adjacent to the underlying surface should be equal to the sum of the descending flux of long-wave radiation and the flux of short-wave radiation absorbed in this layer and by the underlying surface:

$$J^{\uparrow}(m) = s_0 + \int_0^m s(m_1)dm_1 + J^{\downarrow}(m) . \tag{6}$$

Combining this relation with the first one of equations (5) yields:

$$\frac{dJ^{\uparrow}(m)}{dm} = 0.5\, s(m) - 0.5\, w(m) s_0 - 0.5 w(m)\left(\int_0^m s(m_1)\, dm_1\right). \qquad (7)$$

The following is obtained through integrating from m to M_0 and taking into account equality (6):;

$$J^{\uparrow}(m) = s_0 + \int_0^{M_0} s(m_1)dm_1 - 0.5\int_m^{M_0} s(m_1)dm_1 + 0.5\, s_0 \int_m^{M_0} w(m_1)dm_1 + 0.5\int_m^{M_0} w(m_1)\left[\int_0^{m_1} s(m_2)dm_2\right]dm_1$$

$$J^{\downarrow}(m) = 0.5\int_m^{M_0} s(m_1)dm_1 + 0.5\, s_0 \int_m^{M_0} w(m_1)dm_1 + 0.5\int_m^{M_0} w(m_1)\left[\int_0^{m_1} s(m_2)dm_2\right]dm_1; \qquad (8)$$

$$J^{\uparrow}(m) + J^{\downarrow}(m) = s_0 + \int_0^{M_0} s(m_1)dm_1 + s_0 \int_m^{M_0} w(m_1)dm_1 + \int_m^{M_0} w(m_1)\left[\int_0^{m_1} s(m_2)dm_2\right]dm_1$$

The equilibrium profile of air temperature $T_*(m)$ is calculated from the condition that the radiation balance is equal to zero:

$$2\,\sigma\, w(m) T_*^4(m) = w(m)\left(J^{\uparrow}(m) + J^{\downarrow}(m)\right) + s(m);$$

$$T_*(m) = \sqrt[4]{0.5\sigma^{-1}\left(J^{\uparrow}(m) + J^{\downarrow}(m) + \frac{s(m)}{w(m)}\right)}. \qquad (9)$$

Here $\sigma = 5.670367 \cdot 10^{-8}$ W m^{-2} K^{-4} is Stefan's constant.

Since the underlying surface is in a state of radiation equilibrium (see **Introduction**, assumption VI), the temperature of the latter is equal to

$$T_{*0} = \sqrt[4]{\sigma^{-1}(J^{\downarrow}(0) + s_0)}. \qquad (10)$$

If the atmosphere does not absorb short-wave radiation at all ($s(m) \equiv 0$), but receives heat only in the form of long-wave radiation from the underlying surface, formulas (8-10) are substantially simplified:

$$J^{\uparrow}(m) = s_0 + 0.5\, s_0 \int_m^{M_0} w(m_1)dm_1$$

$$J^{\downarrow}(m) = 0.5\, s_0 \int_m^{M_0} w(m_1)dm_1;$$

$$J^{\uparrow}(m) + J^{\downarrow}(m) = s_0\left(1 + \int_m^{M_0} w(m_1)dm_1\right);$$

$$T_*(m) = \sqrt[4]{0.5\sigma^{-1} s_0\left(1 + \int_m^{M_0} w(m_1)dm_1\right)};$$

$$T_{*0} = \sqrt[4]{\sigma^{-1} s_0\left(1 + 0.5\int_0^{M_0} w(m_1)dm_1\right)}$$

Note that $T_{*0} > T_*(0)$. In the real atmosphere, such discontinuities in the temperature field at the boundaries of different physical media are compensated in the course of heat transfer by other, non-radiation processes.

So, in the model (2-4) there is a single equilibrium air temperature profile, characterized by the formula (9). The formulas

$$P_*(m) = g\,(M_0 - m)\,; \tag{11}$$

$$\rho_*(m) = \frac{\mu}{R\,T_*(m)}\,g(M_0 - m) \tag{12}$$

characterize the corresponding vertical profiles of air pressure and density in equilibrium.

Stability of equilibrium

In model (2), equilibrium is established in the course of heat exchange with the help of radiant energy fluxes between the atmospheric layers, as well as between them and the underlying surface. This exchange is carried out at long waves, although the primary source of energy in the model is the flow of short-wave (solar) radiation. It is absorbed by the atmospheric layers ($s(m)$, $m \epsilon[0, M_0]$) and the underlying surface (s_0). In total, the Earth's system in the model absorbs $S_\varepsilon = \left(s_0 + \int_0^{M_0} s(m_1)dm_1\right)$ W m^{-2} of energy in the form of short-wave radiation.

Recall that the only stationary solution $T_*(m)$ of equation (2) is strictly positive, since it satisfies the condition

$$T_*(m) \geq \sqrt[4]{0.5\sigma^{-1}s_0} > 0$$

for all $m \epsilon[0, M_0]$ (see formula (9) in the previous section).

Taking into account the fact that the underlying surface is constantly in a state of radiation equilibrium (see **Introduction**, assumption VI), a layer of air of small mass dm, located at such a height that the mass of air below it in the air column above the surface area of 1 m^2 is equal to m, absorbs during a short time interval dt the following portions of radiant heat (J m^{-2}):

- $s(m)\,dm\,dt$ of short-wave radiation;
- $w(m)s_0\exp\left(-\int_0^m w(m_1)dm_1\right)dm\,dt$ of long-wavelength equivalent of short-wave radiation absorbed by the underlying surface;
- $w(m)\exp\left[-\left|\int_0^m w(m_2)dm_2 - \int_0^{m_1} w(m_2)dm_2\right|\right]\{w(m_1)\,\sigma\,T^4(t,m_1)\,dm_1\}\,dm\,dt$ of long-wave radiation transmitted by the atmosphere from a layer of small mass dm_1, the mass under which is equal to m_1;
- $w(m)\exp\left[-\left(\int_0^m w(m_2)dm_2 + \int_0^{m_1} w(m_2)dm_2\right)\right]\{w(m_1)\,\sigma\,T^4(t,m_1)\,dm_1\}\,dm\,dt$ of long-wave radiation transmitted by the atmosphere from a layer of small mass dm_1 to the underlying surface, emitted by it and transmitted by the atmosphere to a layer of small mass dm, the mass under which is equal to m.

Under these conditions, the process of heat exchange between the atmospheric layers, as well as between them and the underlying surface leads in model (2) to the establishment of an equilibrium vertical distribution of air temperature. Let us deduce this.

Theorem. Let $T(t, m)$ be a continuous function on $[t_0, +\infty]$ x $[0, M_0]$ differentiable in terms of t, $\frac{\partial T(t,m)}{\partial t}$ is also continuous, and $T(t, m)$ satisfies the equation

$$\frac{\partial T(t,m)}{\partial t} = \frac{1}{c_P(m)} \Big\{ w(m) \int_0^{M_0} K(m, m_1)\, w(m_1) \sigma T^4(t, m_1) dm_1 + s(m) + w(m)\, s_0 \exp(-\int_0^m w(m_1) dm_1) - 2\, w(m) \sigma T^4(t, m) \Big\}, \tag{13}$$

where $s_0 > 0$, $w(m)$ and $c_P(m)$ are continuous positive functions on $[0, M_0]$, and

$$K(m, m_1) = \exp[-| \int_0^m w(m_2) dm_2 - \int_0^{m_1} w(m_2) dm_2 |] + \exp[-(\int_0^m w(m_2) dm_2 + \int_0^{m_1} w(m_2) dm_2)] .$$

Then, for any initial condition $T(t_0, m)$ such that $T(t_0, m) \geq 0$, $T(t, m)$ tends uniformly by m to the stationary solution of equation (13) as $t \rightarrow +\infty$.

Proof.

1. Choose C_1 such that .

$$0 < C_1 < \sqrt[4]{0.5\sigma^{-1} s_0 \exp(-\int_0^{M_0} w(m_1) dm_1)} .$$

Since the first two terms in curly brackets on the right-hand side of equation (13) are non-negative, and $w(m)$ and $c_P(m)$ are positive and continuous on $[0, M_0]$, then for $T(t, m) \leq C_1$ the following holds:

$$\frac{\partial T(t,m)}{\partial t} \geq \{\inf_{m_1 \in [0, M_0]} \frac{w(m_1)}{c_P(m_1)}\} \Big\{ s_0 \exp\Big(-\int_0^{M_0} w(m_1) dm_1\Big) - 2\,\sigma\, C_1^4 \Big\} = v > 0.$$

Consequently, in a finite time not greater than C_1/v, for each $m \in [0, M_0]$, the value of $T(t, m)$ becomes greater than C_1. Moreover, subsequently, it cannot return to the limits of the segment $[0, C_1]$, since for $T(t, m) \leq C_1$ the derivative $\frac{\partial T(t,m)}{\partial t}$ exceeds a positive number v. Thus, for $t \geq t_{00} = t_0 + C_1/v$, the inequality $T(t, m) \geq C_1 > 0$ holds for all $m \in [0, M_0]$.

2. Due to condition (13), the following holds for $T(t, m)$ and $T_*(m)$:

$$\frac{\partial T(t,m)}{\partial t} = \frac{1}{c_P(m)} \Big\{ w(m) \int_0^{M_0} K(m, m_1)\, w(m_1)\, \sigma\, T^4(t, m_1) dm_1 + s(m) + w(m)\, s_0 \exp(-\int_0^m w(m_1) dm_1) - 2\, w(m) \sigma T^4(t, m) \Big\};$$

$$\frac{\partial T_*(m)}{\partial t} = \frac{1}{c_P(m)} \Big\{ w(m) \int_0^{M_0} K(m, m_1)\, w(m_1)\, \sigma\, T_*^4\, (m_1) dm_1 + s(m) + w(m)\, s_0 \exp(-\int_0^m w(m_1) dm_1) - 2\, w(m) \sigma\, T_*^4\, (m)) \Big\}.$$

Subtracting the second from the first relation, we obtain:

$$\frac{\partial T(t,m)}{\partial t} = \frac{1}{c_P(m)} \left\{ w(m) \int_0^{M_0} K(m,m_1)\, w(m_1)\, \sigma\, (T^4(t,m_1) - T_*^4\,(m_1))\, dm_1 - 2\, w(m)\, \sigma (T^4(t,m) - T_*^4\,(m)) \right\} =$$
$$\frac{2\,\sigma\, w(m)}{c_P(m)} \left\{ \int_0^{M_0} 0.5\, K(m,m_1)\, w(m_1)(T^4(t,m_1) - T_*^4\,(m_1))\, dm_1 - (T^4(t,m) - T_*^4\,(m)) \right\}.$$

Further, taking into account this relation, we obtain:

$$\frac{\partial (T^4(t,m) - T_*^4\,(m))}{\partial t} = \frac{\partial (T^4(t,m))}{\partial t} = 4\,(T^3(t,m))\, \frac{\partial T(t,\ m)}{\partial t} =$$
$$\frac{8\,\sigma\, w(m)\ T^3(t,m)}{c_P(m)} \left\{ \int_0^{M_0} 0.5\, K(m,m_1)\, w(m_1)\, (T^4(t,m_1) - T_*^4\,(m_1))\, dm_1 - (T^4(t,m) - T_*^4\,(m)) \right\}.$$

3. If, now, in the relation

$$\frac{\partial (T^4(t,m) - T_*^4\,(m))}{\partial t} =$$
$$\frac{8\,\sigma\, w(m)\ T^3(t,m)}{c_P(m)} \left\{ \int_0^{M_0} 0.5\, K(m,m_1)\, w(m_1)\, (T^4(t,m_1) - T_*^4\,(m_1))\, dm_1 - (T^4(t,m) - T_*^4\,(m)) \right\}, \tag{14}$$

one denotes $f(t, m) = T^4(t, m) - T_*^4(m)$, $k(m, m_1) = 0.5\, K(m, m_1)w(m_1)$ and $g(t, m) = \frac{8\,\sigma\, w(m)\ T^3(t,m)}{c_P(m)}$, then for $t \geq t_{00}$ we find ourselves in the conditions of Lemma 2 (see the methodological section).

Indeed, $g\,(t,\ m) = \frac{8\,\sigma\, w(m)\ T^3(t,m)}{c_P(m)} \geq 8\,\sigma\ \{\inf_{m_1 \in [0,M_0]} \frac{w(m_1)}{c_P(m_1)}\}\ C_1^3 = C_0 > 0.$

Moreover, $\int_0^{M_0} k(m,m_1) dm_1 = 0.5 \int_0^{M_0} K(m,m_1)\, w(m_1)\, dm_1 =$

$$1 - 0.5\ \{\exp[-\int_m^{M_0} w(m_2)dm_2] + \exp[-\int_0^m w(m_2)dm_2 - \int_0^{M_0} w(m_2)dm_2]\} =$$

$$1 - 0.5 \exp[-\int_0^{M_0} w(m_2)dm_2]\{\exp[\int_0^m w(m_2)dm_2] + \exp[-\int_0^m w(m_2)dm_2]\}$$

$\leq 1 - \exp\left[-\int_0^{M_0} w(m_2)dm_2\right] = C < 1$.. The last transition uses the fact that the sum of the positive number $\exp[\int_0^m w(m_2)dm_2]$ and the inverse is greater than 2 or equals it.

Thus, by the corollary to Lemma 2, $f(t, m)$ tends to zero as $t \to +\infty$ uniformly by $m \in [0, M_0]$.

■

Conclusions

Modern climate models are widely used for applied purposes. They are often quite complex, describing joint variations in space and time of hundreds and thousands of variables. Of course, studying them for stability is problematic due to their complexity. However, it still needs to be carried out at least in relation to their simplified versions, i.e., so called models of reduced and minimal complexity. Otherwise, there will be no confidence in the adequacy of the applied modeling technology since nonetheless the real climate is undoubtedly stable.

This paper discusses a very particular issue, namely, the stability of a one-dimensional hydrostatic model of a dry atmosphere. It is necessary to conduct a broader study of the equilibrium state of the model and its stability accounting for non-radiative heat transfer, water in various phases and its phase transformations, and also replace the assumption of the radiation equilibrium of the underlying surface with a less limiting one. With the real climate research needs in mind, it would be certainly important to consider a three-dimensional case.

In conclusion, we would like to mention that for the final judgement on the adequacy of hydrostatic models for the description of the evolution of climate, there is still a lack of substantiation of the hydrostatic hypothesis (for example, as an asymptotic case), based on a universal description of dynamics of the atmosphere using the Navier-Stokes equations.

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